UNIVERSIDADE
BEIRA INTERIOR

Elementos Finitos Isoparamétricos

Mecânica Estrutural (10391/10411)

Mestrado Integrado em Engenharia Aeronáutica

2022

Pedro V. Gamboa

Departamento de Ciências Aeroespaciais
Faculdade de Engenharia

1. Definition

The same function that is used to define the element geometry is used to define the displacements within the element.

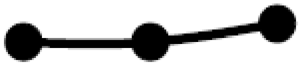
– 2-node truss element

linear geometry

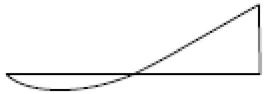
linear displacements

– 3-node beam element

quadratic geometry



quadratic displacements



1. Definition

The same local coordinate system is assigned to each element.

This coordinate system is called the natural coordinate system.

The advantages of choosing this coordinate system are:

- It is easier to define the shape functions
- The integration over the surface of the element is easier (numerical integration can be used which is much simpler in the natural coordinate systems and can be scaled to the actual area)

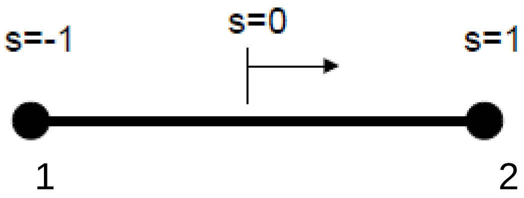
The steps in deriving the elemental stiffness matrices are the same:

- Step 1: Select element type
- Step 2: Select displacement function
- Step 3: Define strain/displacement, stress/strain relations
- Step 4: Derive element stiffness matrix and equations

2. 1D Truss Elements

Step 1 – Select element type

For 1D linear truss elements the natural coordinate system for an element is:



The natural coordinates are related to the global coordinates through

$$x = a_1 + a_2 s \tag{2.1}$$

which can be solved for the a's to give

$$x = \left(\frac{1-s}{2}\right)x_1 + \left(\frac{1+s}{2}\right)x_2 \tag{2.2}$$

2. 1D Truss Elements

or in matrix form as

$$x = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad (2.3)$$

where

$$N_1 = \frac{1-s}{2} \quad ; \quad N_2 = \frac{1+s}{2} \qquad (2.4)$$

Now, following the remainder of the steps becomes much simpler.

Step 2 – Select the displacement function

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (2.5)$$

2. 1D Truss Elements

Step 3 – Define strain/displacement, stress/strain relations

Recall the following definition

$$\varepsilon_x(x) = \frac{du}{dx} \qquad (2.6)$$

Then, by applying the chain rule of differentiation, we have

$$\varepsilon_x(x) = \frac{du}{ds} \Big/ \frac{dx}{ds} \qquad (2.7)$$

Thus

$$\varepsilon_x = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \mathbf{Bu} \qquad (2.8)$$

2. 1D Truss Elements

The stress/strain relation is expressed as

$$\sigma_x = \mathbf{D}\boldsymbol{\varepsilon}_\mathbf{x} \tag{2.9}$$

where **D**=*E*.

Then,

$$\boldsymbol{\sigma}_\mathbf{x} = E\mathbf{B}\mathbf{u} \tag{2.10}$$

Step 4 – Derive the element stiffness matrix and equations

The stiffness matrix is

$$\mathbf{K}^e = \int_L AE\mathbf{B}^\mathrm{T}\mathbf{B}dx \tag{2.11}$$

which has an integral over *x* which needs to be converted to an integral over *s*.

2. 1D Truss Elements

This is done through the transformation

$$\int_0^L f(x)dx = \int_{-1}^1 f(s)|J|ds \tag{2.12}$$

where $|J|$ is the Jacobian and for the simple truss element it is

$$|J| = \frac{dx}{ds} = \frac{L}{2} \tag{2.13}$$

Finally, substituting **B**, $|J|$ and the transformation in (2.12) into (2.11) gives

$$\mathbf{K}^e = AE\int_{-1}^1 \begin{Bmatrix} \frac{-1}{L} \\ \frac{1}{L} \end{Bmatrix} \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \frac{L}{2} ds \tag{2.14}$$

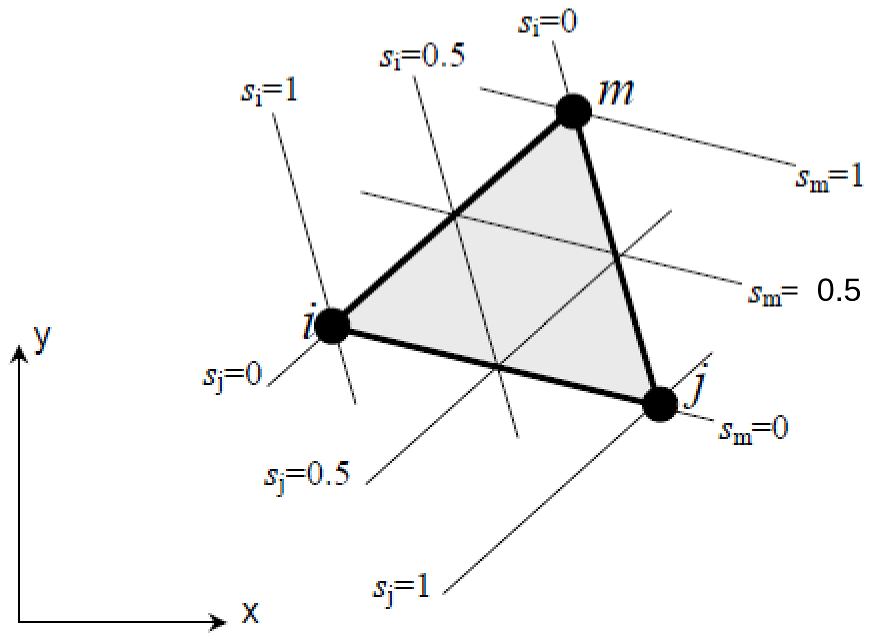
or

$$\mathbf{K}^e = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{2.15}$$

3. CST Elements

Step 1 – Select element type

For constant strain triangular (CST) elements a natural coordinate system as shown in the figure is selected.



3. CST Elements

The geometry is defined in terms of the natural coordinate system as

$$x = s_1 x_i + s_2 x_j + s_3 x_m$$
$$y = s_1 y_i + s_2 y_j + s_3 y_m \qquad (3.1)$$

which in matrix form becomes

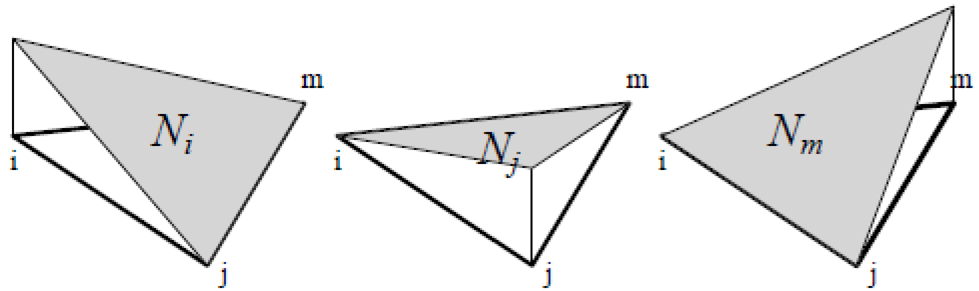
$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_m \\ y_i & y_j & y_m \end{bmatrix} \begin{Bmatrix} s_1 \\ s_2 \\ s_3 \end{Bmatrix} \qquad (3.2)$$

and which can be solved as

$$\begin{Bmatrix} s_1 \\ s_2 \\ s_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_i & x_j & x_m \\ y_i & y_j & y_m \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_j & \beta_j & \gamma_j \\ \alpha_m & \beta_m & \gamma_m \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} \qquad (3.3)$$

3. CST Elements

In this case, these *s*’s are the shape functions

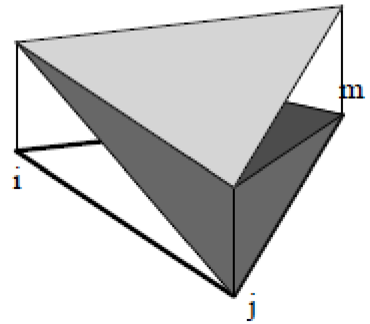


$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ x_j \\ y_j \\ x_m \\ y_m \end{Bmatrix} \quad (3.4)$$

3. CST Elements

The sum of the shape functions anywhere on the element is 1.

$$N_i + N_j + N_m = 1 \tag{3.5}$$



Note that in this case, the *N*'s are simply s_1, s_2 and s_3.

Step 2 – Select the displacement function

The element displacement can be written as a function of the nodal DOF in the same form as used for the geometry.

3. CST Elements

Thus

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \qquad (3.6)$$

or

$$\mathbf{\Psi} = \mathbf{N}\mathbf{u} \qquad (3.7)$$

3. CST Elements

Step 3 – Define strain/displacement, stress/strain relations

In 2D the strain/displacement relations are

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3.6}$$

or in matrix form

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \tag{3.7}$$

3. CST Elements

and

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_m}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_m}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_m}{\partial y} & \frac{\partial N_m}{\partial x} \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \tag{3.8}$$

or

$$\boldsymbol{\varepsilon} = \mathbf{Bu} \tag{3.9}$$

In 2D the stress/strain relations are

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{DBu} \tag{3.10}$$

where D depends on whether plane stress or plain strain exist.

3. CST Elements

Since the shape functions are functions of the natural coordinates s_i and not x and y, the chain rule is applied

$$\begin{aligned}\frac{\partial N_i}{\partial x} &= \frac{\partial N_i}{\partial s_1}\frac{\partial s_1}{\partial x} + \frac{\partial N_i}{\partial s_2}\frac{\partial s_2}{\partial x} + \frac{\partial N_i}{\partial s_2}\frac{\partial s_3}{\partial x} \\ \frac{\partial N_i}{\partial y} &= \frac{\partial N_i}{\partial s_1}\frac{\partial s_1}{\partial y} + \frac{\partial N_i}{\partial s_2}\frac{\partial s_2}{\partial y} + \frac{\partial N_i}{\partial s_2}\frac{\partial s_3}{\partial y}\end{aligned} \tag{3.11}$$

Let us consider the following

$$B_0 = \begin{bmatrix} \frac{\partial N_1}{\partial s_1} & \frac{\partial N_2}{\partial s_1} & \frac{\partial N_3}{\partial s_1} \\ \frac{\partial N_1}{\partial s_2} & \frac{\partial N_2}{\partial s_2} & \frac{\partial N_3}{\partial s_2} \\ \frac{\partial N_1}{\partial s_3} & \frac{\partial N_2}{\partial s_3} & \frac{\partial N_3}{\partial s_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.12}$$

3. CST Elements

Then, for the derivatives of the shape functions with respect to the global coordinate system

$$\frac{\partial N_i}{\partial x} = \frac{\partial s_i}{\partial x} = \frac{\beta_i}{2A} \quad ; \quad \frac{\partial N_j}{\partial x} = \frac{\partial s_j}{\partial x} = \frac{\beta_j}{2A} \quad ; \quad \frac{\partial N_m}{\partial x} = \frac{\partial s_m}{\partial x} = \frac{\beta_m}{2A}$$
$$\frac{\partial N_i}{\partial y} = \frac{\partial s_i}{\partial y} = \frac{\gamma_i}{2A} \quad ; \quad \frac{\partial N_j}{\partial y} = \frac{\partial s_j}{\partial y} = \frac{\gamma_j}{2A} \quad ; \quad \frac{\partial N_m}{\partial y} = \frac{\partial s_m}{\partial y} = \frac{\gamma_m}{2A} \qquad (3.13)$$

and the strains are written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \qquad (3.14)$$

3. CST Elements

Step 4 – Derive element stiffness matrix and equations

Lastly, using the principle of minimum potential energy/principal of virtual work (PMPE) to obtain the stiffness matrix in the form

$$\iiint_V \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{u}\,dV - \mathbf{P} - \iiint_V \mathbf{N}^{\mathrm{T}}\mathbf{X}_{body}\,dV - \iint_S \mathbf{N}^{\mathrm{T}}T_{tract}\,dS = 0 \qquad (3.15)$$

Since all the terms in **B** are constant and assuming the thickness *t* and material properties are constant over the element one has

$$\mathbf{K}\mathbf{u} = \mathbf{f} \qquad (3.16)$$

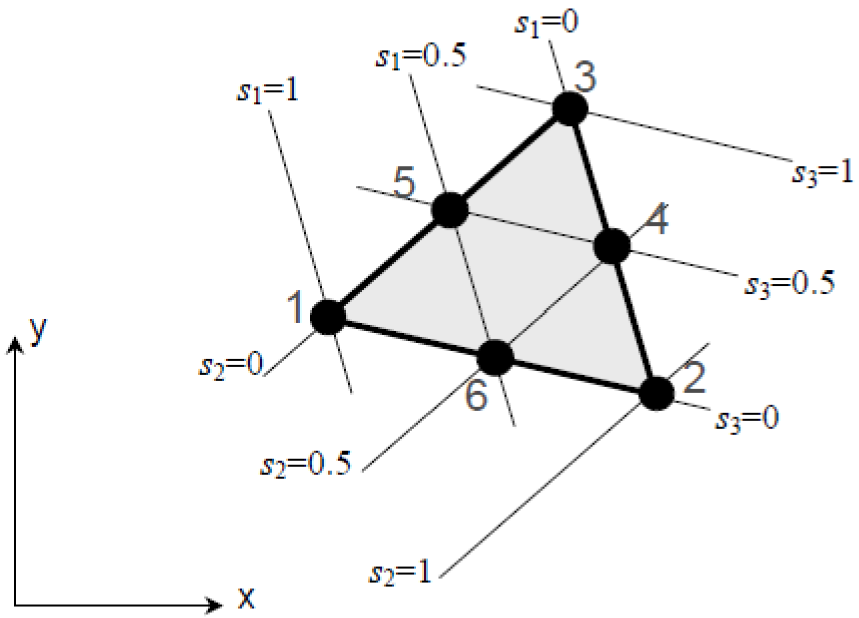
where

$$\mathbf{K} = tA\mathbf{B}^T\mathbf{D}\mathbf{B} \qquad (3.17)$$

4. LST Elements

Step 1 – Select element type

For linear strain triangular (LST) elements a natural coordinate system as shown in the figure is selected similar to the CST element.



4. LST Elements

The geometry is defined in terms of the natural coordinate system as

$$
\begin{aligned}
x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 + N_5 x_5 + N_6 x_6 \\
y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 + N_5 y_5 + N_6 y_6
\end{aligned}
\tag{4.1}
$$

These equations can be solved for the shape functions in terms of the natural coordinates.

Let N_i be a quadratic function of s_1 and s_2 of the form

$$
N_i = a_{0i} + a_{1i} s_1 + a_{2i} s_2 + a_{3i} s_1^2 + a_{4i} s_2^2 + a_{5i} s_1 s_2 \tag{4.2}
$$

which means that there are 6 unknown coefficients to be determined for each shape function.

Note that it is possible to express s_3 as a function of s_1 and s_2 as

$$
s_3 = 1 - s_1 - s_2
$$

4. LST Elements

Using the information that at node *i* we need N_i=1 and all other $N_{j\neq i}$=0, then there are 6 equations for each shape function and we can solve for the coefficients.

Then

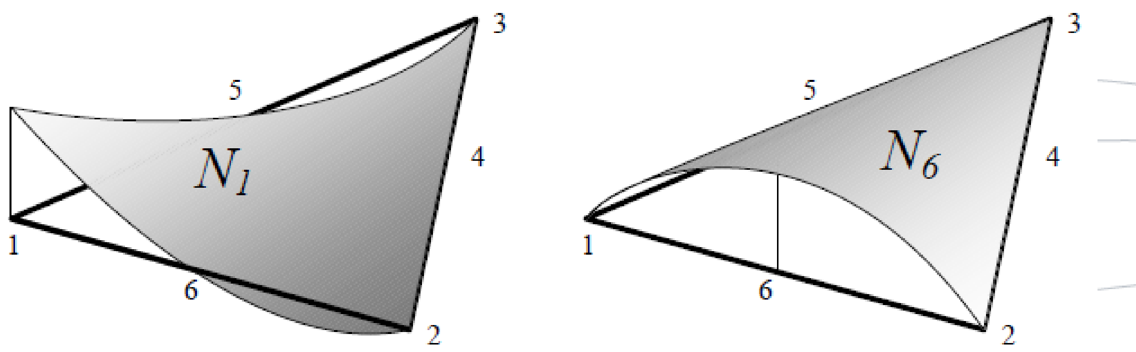
$$\begin{aligned} N_1 &= s_1(2s_1-1) \\ N_2 &= s_2(2s_2-1) \\ N_3 &= s_3(2s_3-1) \\ N_4 &= 4s_2s_3 \\ N_5 &= 4s_3s_1 \\ N_6 &= 4s_1s_2 \end{aligned} \qquad (4.3)$$

4. LST Elements

Noting that $s_3 = 1 - s_1 - s_2$, then (4.3) becomes

$$
\begin{aligned}
N_1 &= s_1(2s_1 - 1) \\
N_2 &= s_2(2s_2 - 1) \\
N_3 &= (1 - s_2 - s_2)[2(1 - s_2 - s_2) - 1] \\
N_4 &= 4s_2(1 - s_2 - s_2) \\
N_5 &= 4(1 - s_2 - s_2)s_1 \\
N_6 &= 4s_1 s_2
\end{aligned}
\qquad (4.4)
$$

In this case the shape/interpolation functions look like



4. LST Elements

The sum of the shape functions anywhere on the element is 1.

$$N_1 + N_2 + N_3 + N_4 + N_5 + N_6 = 1 \tag{4.5}$$

Incidentally, the shape functions in the global coordinate system for a nice element with sides aligned with the *x* and *y* axes would look something like this

$$N_1 = 1 - 3x/b - 3y/h + 2x^2/b^2 + 4xy/(bh) + 2y^2/h^2$$

$$N_2 = -x/b + 2x^2/b^2$$

$$N_3 = -y/h + 2y^2/h^2$$

$$N_4 = 4xy/(bh)$$

$$N_5 = 4y/h - 4xy/(bh) + 4y^2/h^2$$

$$N_6 = 4x/b - 4xy/(bh) - 4x^2/b^2$$

This is why the isoparametric formulation is used.

4. LST Elements

Step 2 – Select the displacement function

The element displacement can be written as a function of the nodal DOF in the same form as used to describe the geometry.

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 \end{bmatrix}$$
$$\times \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4 \quad u_5 \quad v_5 \quad u_6 \quad v_6\}^T \tag{4.6}$$

or

$$\mathbf{\Psi} = \mathbf{N}\mathbf{u} \tag{4.7}$$

4. LST Elements

Step 3 – Define strain/displacement, stress/strain relations

In 2D the strain/displacement relations are

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{4.8}$$

or in matrix form

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{4.9}$$

4. LST Elements

In 2D the stress/strain relations are

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{DBu} \tag{4.10}$$

where D depends on whether plane stress or plain strain exist.

Let us consider the following matrix

$$B_0 = \begin{bmatrix} \frac{\partial N_1}{\partial s_1} & \frac{\partial N_2}{\partial s_1} & \frac{\partial N_3}{\partial s_1} & \frac{\partial N_4}{\partial s_1} & \frac{\partial N_5}{\partial s_1} & \frac{\partial N_6}{\partial s_1} \\ \frac{\partial N_1}{\partial s_2} & \frac{\partial N_2}{\partial s_2} & \frac{\partial N_3}{\partial s_2} & \frac{\partial N_4}{\partial s_2} & \frac{\partial N_5}{\partial s_2} & \frac{\partial N_6}{\partial s_2} \end{bmatrix} \tag{4.11}$$

$$= \begin{bmatrix} 4s_1 - 1 & 0 & 4s_1 + 4s_2 - 3 & -4s_2 & 4 - 8s_1 - 4s_2 & 4s_2 \\ 0 & 4s_1 - 1 & 4s_1 + 4s_2 - 3 & 4 - 4s_1 - 8s_2 & -4s_1 & 4s_1 \end{bmatrix}$$

4. LST Elements

Let the Jacobian matrix be (a 2x2 matrix)

$$J = \begin{bmatrix} \frac{\partial N_1}{\partial s_1} & \frac{\partial N_2}{\partial s_1} & \frac{\partial N_3}{\partial s_1} & \frac{\partial N_4}{\partial s_1} & \frac{\partial N_5}{\partial s_1} & \frac{\partial N_6}{\partial s_1} \\ \frac{\partial N_1}{\partial s_2} & \frac{\partial N_2}{\partial s_2} & \frac{\partial N_3}{\partial s_2} & \frac{\partial N_4}{\partial s_2} & \frac{\partial N_5}{\partial s_2} & \frac{\partial N_6}{\partial s_2} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \end{bmatrix} \qquad (4.12)$$

Then, the terms in the *B* matrix are extracted from the product

$$J^{-1} B_0 = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y} & \frac{\partial N_5}{\partial y} & \frac{\partial N_6}{\partial y} \end{bmatrix} \qquad (4.13)$$

4. LST Elements

Step 4 – Derive element stiffness matrix and equations

Lastly, using the principle of minimum potential energy/principal of virtual work (PMPE) to obtain the stiffness matrix in the form

$$\iiint_V \mathbf{B}^{\mathbf{T}}\mathbf{D}\mathbf{B}\mathbf{u}\,dV - \mathbf{P} - \iiint_V \mathbf{N}^{\mathbf{T}}\mathbf{X}_{body}\,dV - \iint_S \mathbf{N}^{\mathbf{T}}T_{tract}\,dS = 0 \qquad (4.14)$$

The Gaussian Quadrature is used to perform the integration over the element.

Note that B and N in the above are functions of the natural coordinates s_1 and s_2.

5. Numerical integration using Gaussian Quadrature

As seen before, the derivation of the stiffness matrix requires an integration over the element from the strain energy definition and so does the force vector.

Often this is difficult to do explicitly. Some numerical integration techniques can help here.

In the element formulation, the form of the displacement function has been chosen and hence the form of the strains and stresses which appear in the internal strain energy.

The principle behind the Gaussian Quadrature is that if the functional form of the function to be integrated is known, then there is a certain number of points where the function needs to be evaluated which will give an exact representation of the integral.

5. Numerical integration using Gaussian Quadrature

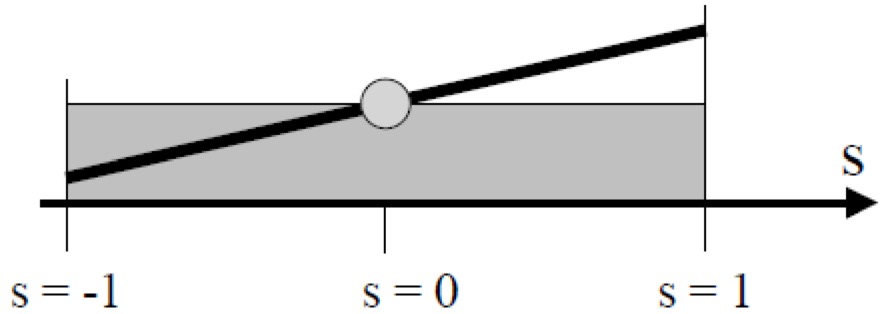
The Gauss formula is

$$I = \int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} W_i f(x_i) \tag{5.1}$$

The integral is evaluated by calculating the function at discrete points x_i and multiplying it by an appropriate weight W_i.

For an n integration point rule then the accuracy order is $2n$-1.

For example, 1 integation point will integrate a first order polynomial exactly:



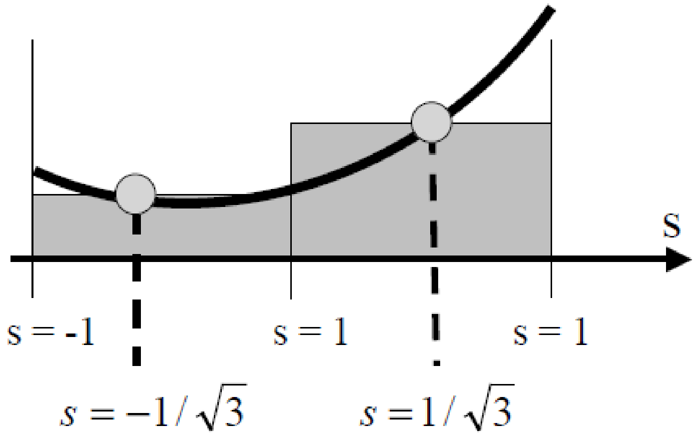
$n = 1$

$s_1 = 0$

$W_1 = 2$

5. Numerical integration using Gaussian Quadrature

For example, 2 integation points will integrate a second order polynomial exactly:



$$n = 2$$
$$s_{1,2} = \pm 1/\sqrt{3}$$
$$W_1 = W_2 = 1$$

The Gauss formula in two dimensions is

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt = \int_{-1}^{1}\left[\sum_{i=1} W_i f(s_i,t)\right]dt$$

$$= \left\{\sum_{j=1} W_j \left[\sum_{i=1} W_i f(s_i,t)\right]\right\}\Bigg|_j = \sum_i \sum_j W_i W_j f(s_i,t_j) \qquad (5.2)$$

5.1. Example for CST element

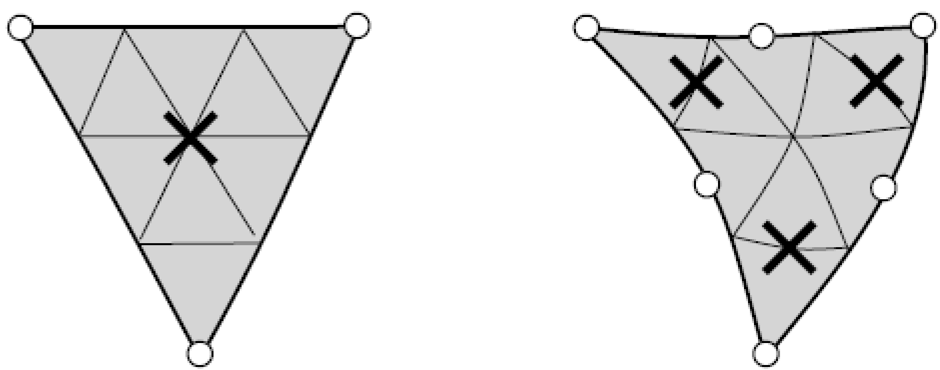
For the CST element the displacement function is linear, resulting in constant strain and stress fields over the element.

To find the integral of a constant, i.e., the area under the curve, it is only necessary to evaluate the function at one point.

For a CST element, this point is located at the center of the triangle.

In the natural coordinate system, this point is located at $s_1=s_2=s_3=0.333$ and the corresponding weight is 1.

For example, 3-node versus 6-node triangular elements:



5.2. Example for LST element

For the LST element, **B** and **N** have been expressed in terms of the natural coordinates.

For this element, there are 3 Gauss points with locations and weights:

$$
\begin{aligned}
&group1 \quad s_2 = s_3 = 0.1666, \quad s_1 = 0.666, \quad W_1 = 0.333 \\
&group2 \quad s_1 = s_3 = 0.1666, \quad s_2 = 0.666, \quad W_2 = 0.333 \\
&group3 \quad s_1 = s_2 = 0.1666, \quad s_3 = 0.666, \quad W_3 = 0.333
\end{aligned}
$$

This gives a degree of precision of 2 (integrates a second order polynomial exactly).

5.2. Example for LST element

The stiffness matrix becomes

$$k^2 = t\iint_{x-y} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\, dA_{x-y} = t\iint_{s_1-s_2} \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}|\mathbf{J}|\, dA_{s_i} = \frac{t}{2}\sum_{i=1}^{n} W_i \left(\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\right)_i |\mathbf{J}|_i$$

$$= t\frac{0.33}{2}\left(\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}|\mathbf{J}|\right)_{group1} + t\frac{0.33}{2}\left(\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}|\mathbf{J}|\right)_{group2} + t\frac{0.33}{2}\left(\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}|\mathbf{J}|\right)_{group3} \quad (5.3)$$

where the Jacobian has already been developed when matrix **B** was created.

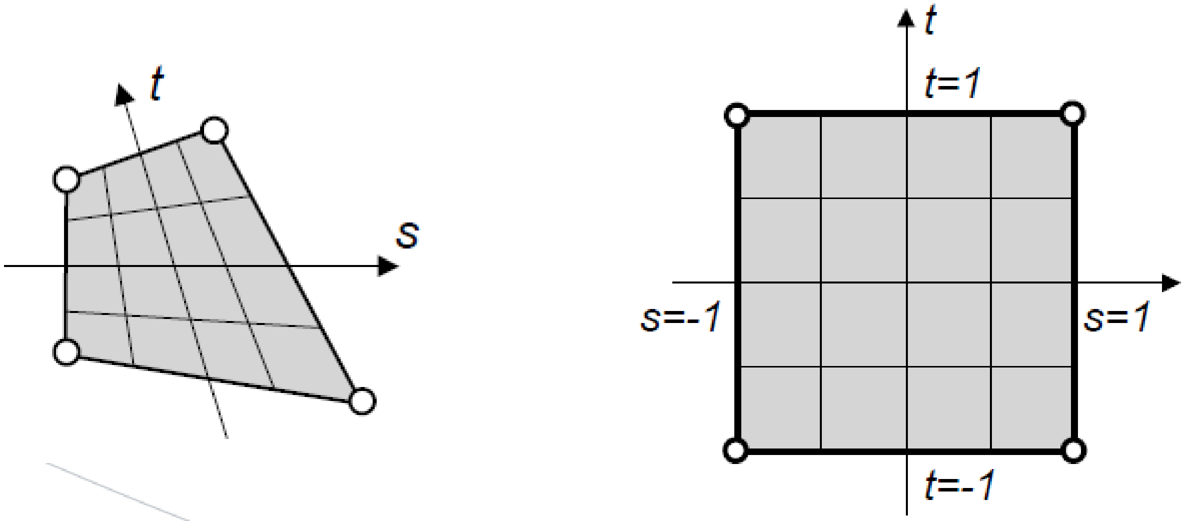
The integral for the force vector is derived in the same way.

Note: the factor of ½ comes from the area of the triangle in the s_1-s_2 space

6. 4-node quadrilateral element

Step 1 – Select element type

The natural coordinate system shown in the figure is selected.



The geometry in terms of the natural coordinate system is

$$x = a_1 + a_2 s + a_3 t + a_4 st$$
$$y = b_1 + b_2 s + b_3 t + b_4 st \qquad (6.1)$$

6. 4-node quadrilateral element

or in terms of the shape functions and the nodal coordinates as

$$\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 \end{aligned} \qquad (6.2)$$

which can be written in matrix form as

$$\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \qquad (6.3)$$

6. 4-node quadrilateral element

The shape functions are

$$
\begin{aligned}
N_1 &= \frac{1}{4}(1-s)(1-t) \\
N_2 &= \frac{1}{4}(1+s)(1-t) \\
N_3 &= \frac{1}{4}(1+s)(1+t) \\
N_4 &= \frac{1}{4}(1-s)(1+t)
\end{aligned}
\qquad (6.4)
$$

The sum of the shape functions anywhere on the element is 1.

6. 4-node quadrilateral element

Thus

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix} \quad (6.5)$$

6. 4-node quadrilateral element

Step 2 – Select the displacement function

The element displacement can be written as a function of the nodal DOF in the same form as used to describe the geometry.

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \tag{6.6}$$

or

$$\boldsymbol{\Psi} = \mathbf{N}\mathbf{u} \tag{6.7}$$

6. 4-node quadrilateral element

Step 3 – Define strain/displacement, stress/strain relations

In 2D the strain/displacement relations are

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{6.8}$$

or in matrix form

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \tag{6.9}$$

6. 4-node quadrilateral element

or

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

6. 4-node quadrilateral element

and

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad (6.10)$$

That is

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{u}$$

6. 4-node quadrilateral element

In 2D the stress/strain relations are

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{DBu}$$

where D depends on whether plane stress or plain strain exist. But N_i are functions of s and t, not x and y, so, applying the chain rule we have

$$\begin{aligned}\frac{\partial N_i}{\partial x} &= \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial N_i}{\partial t}\frac{\partial t}{\partial x}\\ \frac{\partial N_i}{\partial y} &= \frac{\partial N_i}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial N_i}{\partial t}\frac{\partial t}{\partial y}\end{aligned} \qquad (6.11)$$

or in matrix form

$$\begin{Bmatrix}\dfrac{\partial N_i}{\partial x}\\ \dfrac{\partial N_i}{\partial y}\end{Bmatrix} = \begin{bmatrix}\dfrac{\partial s}{\partial x} & \dfrac{\partial t}{\partial x}\\ \dfrac{\partial s}{\partial y} & \dfrac{\partial t}{\partial y}\end{bmatrix}\begin{Bmatrix}\dfrac{\partial N_i}{\partial s}\\ \dfrac{\partial N_i}{\partial t}\end{Bmatrix} \qquad (6.12)$$

6. 4-node quadrilateral element

The derivatives

$$\frac{\partial s}{\partial x}, \frac{\partial t}{\partial x}, \frac{\partial s}{\partial y}, \frac{\partial t}{\partial y}$$

are dificult to evaluate but the derivatives

$$\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}$$

are not.

Then, it is possible to write

$$\begin{aligned} \frac{\partial N_i}{\partial x} &= \frac{1}{|J|}\left(\frac{\partial N_i}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial N_i}{\partial t}\frac{\partial y}{\partial s}\right) \\ \frac{\partial N_i}{\partial y} &= \frac{1}{|J|}\left(-\frac{\partial N_i}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial N_i}{\partial t}\frac{\partial x}{\partial s}\right) \end{aligned} \tag{6.13}$$

6. 4-node quadrilateral element

where the determinant of the jacobian, $|J|$, is

$$|J| = \det\begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial s} & \dfrac{\partial y}{\partial t} \end{bmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \qquad (6.14)$$

Now we get a new **B** which is equal to **B** but is a function of *s* and *t*.

6. 4-node quadrilateral element

Step 4 – Derive element stiffness matrix and equations

Lastly, using the principle of minimum potential energy/principal of virtual work (PMPE) to obtain the stiffness matrix in the form

$$t\iint_A \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\mathbf{u}\,dA - \mathbf{P} - \iiint_V \mathbf{N}^{\mathrm{T}}\mathbf{X}_{body}\,dV - \iint_S \mathbf{N}^{\mathrm{T}}T_{tract}\,dS = 0 \qquad (6.15)$$

The Gaussian Quadrature is used to perform the integration over the element.

The intgrals in the *x*-*y* plane are transformed to integrals in the *s*-*t* plane from -1 to +1 and the Gaussian Quadrature is used

$$\int_A f(x)\,dxdy = \int_{-1}^{1}\int_{-1}^{1} f(s)\left|J\right|dsdtt = 4\sum_{i=1}^{n} W_i\left(\mathbf{B}^T\mathbf{D}\mathbf{B}\right)_i\left|\mathbf{J}\right|_i \qquad (6.16)$$